Your real estate agent tells you the property tax on a $170,000 home in a certain town is $2550. Suppose you are about to purchase a home in this town for $190,000. We can use variation to estimate the property tax on the home you are about to purchase. In this section, we will discuss four different types of variation and show how variation can be used to solve real-life problems.

Direct Variation

Many scientific formulas are expressed as variations. A variation is an equation that relates one variable to one or more other variables through the operations of multiplication or division (or both operations). There are essentially four types of variation problems: direct, inverse, joint, and combined variation.

In direct variation, the values of the two related variables increase together or decrease together; that is, as one increases so does the other, and as one decreases so does the other.

Consider a car traveling at 40 miles an hour. The car travels 40 miles in 1 hour, 80 miles in 2 hours, and 120 miles in 3 hours. Note that, as the time increases, the distance traveled increases, and, as the time decreases, the distance traveled decreases.

The formula used to calculate distance traveled is

$$ \text{Distance} = \text{rate} \cdot \text{time} $$

Since the rate is a constant 40 miles per hour, the formula can be written

$$ d = 40t $$

We say that distance varies directly as time or that distance is directly proportional to time.

The preceding equation is an example of direct variation.

DIRECT VARIATION

If a variable $y$ varies directly with a variable $x$, then

$$ y = kx $$

where $k$ is the constant of proportionality (or the variation constant).

Examples 1 through 4 illustrate direct variation.

EXAMPLE 1 Direct Variation in Physics

The length that a spring will stretch, $S$, varies directly with the force (or weight) $F$, attached to the spring. Write the equation for the length that a spring will stretch, $S$, if the constant of proportionality is 0.07.

SOLUTION

$$ S = kF \quad S \text{ varies directly as } F. $$

$$ S = 0.07F \quad \text{Constant of proportionality is } 0.07. $$
**EXAMPLE 2  Direct Variation in Medicine**

The recommended dosage, $d$, of the antibiotic drug vancomycin is directly proportional to a person’s weight, $w$.

a) Write this variation as an equation.
b) Find the recommended dosage, in milligrams, for Doug Kulzer, who weighs 192 lb. Assume the constant of proportionality for the dosage is 18.

**SOLUTION**

a) $d = kw$
b) $d = 18(192) = 3456$

The recommended dosage for Doug Kulzer is 3456 mg.

In certain variation problems, the constant of proportionality, $k$, may not be known. In such cases, we can often find it by substituting the given values in the variation formula and solving for $k$.

**EXAMPLE 3  Finding the Constant of Proportionality**

Suppose $w$ varies directly as the square of $y$. If $w$ is 60 when $y$ is 20, find the constant of proportionality.

**SOLUTION**

Since $w$ varies directly as the square of $y$, we begin with the formula $w = ky^2$. Since the constant of proportionality is not given, we must find $k$ using the given information. Substitute 60 for $w$ and 20 for $y$.

\[
\begin{align*}
  w &= ky^2 \\
  60 &= k(20)^2 \\
  60 &= 400k \\
  \frac{60}{400} &= \frac{400k}{400} \\
  0.15 &= k
\end{align*}
\]

Thus, the constant of proportionality is 0.15.

**EXAMPLE 4  Using the Constant of Proportionality**

The area, $a$, of a picture projected on a movie screen varies directly as the square of the distance, $d$, from the projector to the screen. If a projector at a distance of 25 feet projects a picture with an area of 100 square feet, what is the area of the projected picture when the projector is at a distance of 40 feet?

**SOLUTION**

We begin with the formula $a = kd^2$. Since the constant of proportionality is not given, we must determine $k$, using the given information.

\[
\begin{align*}
  a &= kd^2 \\
  100 &= k(25)^2 \\
  100 &= k(625) \\
  \frac{100}{625} &= k \\
  0.16 &= k
\end{align*}
\]
We now use $k = 0.16$ to determine $a$ when $d = 40$.

$$a = kd^2$$

$$a = 0.16d^2$$

$$a = 0.16(40)^2$$

$$a = 0.16(1600)$$

$$a = 256 \text{ ft}^2$$

Thus, the area of a projected picture is 256 ft$^2$ when the projector is at a distance of 40 ft.

**Inverse Variation**

A second type of variation is *inverse variation*. When two quantities vary inversely, as one quantity increases the other quantity decreases, and vice versa.

To explain inverse variation, we use the formula, distance $=$ rate $\cdot$ time. If we solve for time, we get time $=$ distance/rate. Assume the distance is fixed at 100 miles; then

$$\text{Time} = \frac{100}{\text{rate}}$$

At 100 miles per hour, it would take 1 hour to cover this distance. At a rate of 50 miles an hour, it would take 2 hours. At a rate of 25 miles an hour, it would take 4 hours. Note that as the rate (or speed) decreases, the time increases and vice versa.

The preceding equation can be written

$$t = \frac{100}{r}$$

This equation is an example of an inverse variation. The time and rate are inversely proportional. The constant of proportionality in this case is 100.

**INVERSE VARIATION**

If a variable $y$ varies inversely with a variable $x$, then

$$y = \frac{k}{x}$$

where $k$ is the constant of proportionality.

Two quantities vary inversely, or are *inversely proportional*, when as one quantity increases the other quantity decreases and vice versa. Examples 5 and 6 illustrate inverse variation.
EXAMPLE 5 Inverse Variation in Astronomy

The velocity, \( v \), of a meteor approaching Earth varies inversely as the square root of its distance from the center of Earth. Assuming the velocity is 2 miles per second at a distance, \( d \), of 6400 miles from the center of Earth, determine the equation that expresses the relationship between the velocity of a meteor and its distance from the center of Earth.

**SOLUTION** Since the velocity of the meteor varies inversely as the *square root* of its distance from the center of Earth, the general form of the equation is

\[
v = \frac{k}{\sqrt{d}}
\]

To determine \( k \), we substitute the given values for \( v \) and \( d \).

\[
2 = \frac{k}{\sqrt{6400}}
\]

\[
2 = \frac{k}{80}
\]

\[
(2)(80) = k
\]

\[
160 = k
\]

Thus, the formula is \( v = \frac{160}{\sqrt{d}} \).

EXAMPLE 6 Using the Constant of Proportionality

Suppose \( y \) varies inversely as \( x \). If \( y = 8 \) when \( x = 15 \), find \( y \) when \( x = 18 \).

**SOLUTION** First write the inverse variation, then solve for \( k \).

\[
y = \frac{k}{x}
\]

\[
8 = \frac{k}{15}
\]

\[
120 = k
\]

Now substitute 120 for \( k \) in \( y = \frac{k}{x} \) and find \( y \) when \( x = 18 \).

\[
y = \frac{120}{18} = \frac{120}{18} = 6.7 \quad \text{(to the nearest tenth)}
\]

Joint Variation

One quantity may vary directly as a product of two or more other quantities. This type of variation is called *joint variation*.
JOINT VARIATION
The general form of a joint variation, where \(y\) varies directly as \(x\) and \(z\), is

\[ y = kxz \]

where \(k\) is the constant of proportionality.

EXAMPLE 7 Joint Variation in Geometry
The area, \(A\), of a triangle varies jointly as its base, \(b\), and height, \(h\). If the area of a triangle is 48 in.\(^2\) when its base is 12 in. and its height is 8 in., find the area of a triangle whose base is 15 in. and whose height is 20 in.

**SOLUTION** First write the joint variation, then substitute the known values and solve for \(k\).

\[ A = kbh \]
\[ 48 = k(12)(8) \]
\[ 48 = k(96) \]
\[ \frac{48}{96} = k \]
\[ \frac{1}{2} = k \]

Now solve for the area of the given triangle.

\[ A = kbh \]
\[ = \frac{1}{2}(15)(20) \]
\[ = 150 \text{ in.}^2 \]

SUMMARY OF VARIATIONS

<table>
<thead>
<tr>
<th>Direct</th>
<th>Inverse</th>
<th>Joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = kx)</td>
<td>(y = \frac{k}{x})</td>
<td>(y = kxz)</td>
</tr>
</tbody>
</table>

Combined Variation

Often in real-life situations, one variable varies as a combination of variables. The following examples illustrate the use of combined variations.

EXAMPLE 6 Combined Variation in Engineering
The load, \(L\), that a horizontal beam can safely support varies jointly as the width, \(w\), and the square of the depth, \(d\), and inversely as the length, \(l\). Express \(L\) in terms of \(w\), \(d\), \(l\), and the constant of proportionality, \(k\).

**SOLUTION**

\[ L = \frac{kwd^2}{l} \]
EXAMPLE 9  **Hot Dog Price, Combined Variation**

The owners of Henrietta Hots find their weekly sales of hot dogs, $S$, vary directly with their advertising budget, $A$, and inversely with their hot dog price, $P$. When their advertising budget is $600 and the price of a hot dog is $1.50, they sell 5600 hot dogs a week.

a) Write a variation expressing $S$ in terms of $A$ and $P$. Include the value of the constant of proportionality.

b) Find the expected sales if the advertising budget is $800 and the hot dog price is $1.75.

**SOLUTION**

a) Since $S$ varies directly as $A$ and inversely as $P$, we begin with the equation

$$S = \frac{kA}{P}$$

We now find $k$ using the known values.

$$5600 = \frac{k(600)}{1.50}$$

$$5600 = 400k$$

$$14 = k$$

Therefore, the equation for the weekly sales of hot dogs is $S = \frac{14A}{P}$.

b) $$S = \frac{14A}{P}$$

$$\begin{align*}
&= \frac{14(800)}{1.75} \\
&= 6400
\end{align*}$$

Henrietta Hots can expect to sell 6400 hot dogs a week if the advertising budget is $800 and the hot dog price is $1.75.

EXAMPLE 10  **Combined Variation**

$A$ varies jointly as $B$ and $C$ and inversely as the square of $D$. If $A = 1$ when $B = 9$, $C = 4$, and $D = 6$, find $A$ when $B = 8$, $C = 12$, and $D = 5$.

**SOLUTION**

We begin with the equation

$$A = \frac{kBC}{D^2}$$

We must first find the constant of proportionality, $k$, by substituting the known values for $A$, $B$, $C$, and $D$ and solving for $k$.

$$1 = \frac{k(9)(4)}{6^2}$$
Thus, the constant of proportionality equals 1. Now we find $A$ for the corresponding values of $B$, $C$, and $D$.

$$1 = \frac{36k}{36}$$

$$1 = k$$

$$A = \frac{kBC}{D^2}$$

$$A = \frac{(1)(8)(12)}{5^2} = \frac{96}{25} = 3.84$$

---

**SECTION 4.1 EXERCISES**

**CONCEPT/Writing Exercises**

*In Exercises 1–4, use complete sentences to answer the question.*

1. Describe direct variation.
2. Describe inverse variation.
3. Describe joint variation.
4. Describe combined variation.

*In Exercises 5–20, use your intuition to determine whether the variation between the indicated quantities is direct or inverse.*

5. The distance between two cities on a map and the actual distance between the two cities
6. The time required to fill a pool with a hose and the volume of water coming from the hose
7. The time required to boil water on a burner and the temperature of the burner
8. A person’s salary and the amount of money withheld from his or her salary for federal income taxes
9. The interest earned on an investment and the interest rate
10. The volume of a balloon and its radius
11. A person’s speed and the time needed for the person to complete the race
12. The time required to cool a room and the temperature of the room
13. The number of workers hired to install a fence and the time required to install the fence
14. The number of calories in a slice of pizza and the size of the slice
15. The time required to defrost a frozen hamburger in a room and the temperature of the room
16. On Earth, the weight and mass of an object
17. The number of people in line at a bank and the amount of time required for the last person to reach the teller

18. The number of books that can be placed upright on a shelf 3 ft long and the width of the books

19. The amount of fertilizer needed to fertilize a lawn and the area of the lawn

20. The percent of light that filters through water and the depth of the water

*In Exercises 21 and 22, use Exercises 5–20 as a guide.*

21. Name two items that have not been mentioned in this section that have a direct variation.

22. Name two items that have not been mentioned in this section that have an inverse variation.

**PRACTICE THE SKILLS**

*In Exercises 23–40, (a) write the variation and (b) determine the quantity indicated.*

23. \( y \) varies directly as \( x \). Determine \( y \) when \( x = 15 \) and \( k = 8 \).

24. \( x \) varies inversely as \( y \). Determine \( x \) when \( y = 7 \) and \( k = 14 \).

25. \( m \) varies inversely as the square of \( n \). Determine \( m \) when \( n = 8 \) and \( k = 16 \).

26. \( r \) varies directly as the square of \( s \). Determine \( r \) when \( s = 2 \) and \( k = 13 \).

27. \( A \) varies directly as \( B \) and inversely as \( C \). Determine \( A \) when \( B = 5 \), \( C = 10 \), and \( k = 5 \).

28. \( M \) varies directly as \( J \) and inversely as \( C \). Determine \( M \) when \( J = 10 \), \( C = 20 \), and \( k = 8 \).

29. \( F \) varies jointly as \( D \) and \( E \). Determine \( F \) when \( D = 3 \), \( E = 10 \), and \( k = 7 \).

30. \( A \) varies jointly as \( R_1 \) and \( R_2 \) and inversely as the square of \( L \). Determine \( A \) when \( R_1 = 120 \), \( R_2 = 8 \), \( L = 5 \), and \( k = \frac{1}{2} \).

31. \( t \) varies directly as the square of \( d \) and inversely as \( f \). If \( t = 192 \) when \( d = 8 \) and \( f = 4 \), determine \( t \) when \( d = 10 \) and \( f = 6 \).

32. \( y \) varies directly as the square root of \( t \) and inversely as \( s \). If \( y = 12 \) when \( t = 36 \) and \( s = 2 \), determine \( y \) when \( t = 81 \) and \( s = 4 \).

33. \( Z \) varies jointly as \( W \) and \( Y \). If \( Z = 12 \) when \( W = 9 \) and \( Y = 4 \), determine \( Z \) when \( W = 50 \) and \( Y = 6 \).

34. \( y \) varies directly as the square of \( R \). If \( y = 4 \) when \( R = 4 \), determine \( y \) when \( R = 8 \).

35. \( H \) varies directly as \( L \). If \( H = 15 \) when \( L = 50 \), determine \( H \) when \( L = 10 \).

36. \( C \) varies inversely as \( J \). If \( C = 7 \) when \( J = 0.7 \), determine \( C \) when \( J = 12 \).

37. \( A \) varies directly as the square of \( B \). If \( A = 245 \) when \( B = 7 \), determine \( A \) when \( B = 12 \).

38. \( F \) varies jointly as \( M_1 \) and \( M_2 \) and inversely as the square of \( d \). If \( F = 20 \) when \( M_1 = 5 \), \( M_2 = 10 \), and \( d = 0.2 \), determine \( F \) when \( M_1 = 10 \), \( M_2 = 20 \), and \( d = 0.4 \).

39. \( F \) varies jointly as \( q_1 \) and \( q_2 \) and inversely as the square of \( d \). If \( F = 80 \) when \( q_1 = 4 \), \( q_2 = 16 \), and \( d = 0.4 \), determine \( F \) when \( q_1 = 12 \), \( q_2 = 20 \), and \( d = 0.2 \).

40. \( S \) varies jointly as \( I \) and the square of \( T \). If \( S = 4 \) when \( I = 10 \) and \( T = 4 \), determine \( S \) when \( I = 4 \) and \( T = 6 \).

**PROBLEM SOLVING**

*In Exercises 41–49, (a) write the variation and (b) determine the quantity indicated.*

41. **Property Tax** The property tax, \( T \), on a home is directly proportional to the assessed value, \( v \), of the home. If the property tax on a home with an assessed value of $140,000 is $2100, what is the property tax on a home with an assessed value of $180,000?

42. **Resistance** The resistance, \( R \), of a wire varies directly as its length, \( L \). If the resistance of a 30 ft length of wire is 0.24 ohm, determine the resistance of a 40 ft length of wire.
43. **Speaker Loudness** The loudness of a stereo speaker, $l$, measured in decibels (dB), is inversely proportional to the square of the distance, $d$, of the listener from the speaker. If the loudness is 20 dB when the listener is 6 ft from the speaker, what is the loudness when the listener is 3 ft from the speaker?

44. **Melting an Ice Cube** The time, $t$, for an ice cube to melt is inversely proportional to the temperature, $T$, of the water in which the ice cube is placed. If it takes an ice cube 2 minutes to melt in $75^\circ$F water, how long will it take an ice cube of the same size to melt in $80^\circ$F water?

45. **Video Rentals** The number of weekly videotape rentals, $R$, at Busterblock Video varies directly with the advertising budget, $A$, and inversely with the daily rental price, $P$. When the video store’s advertising budget is $600 and the rental price is $3 per day, it rents 4800 tapes per week. How many tapes would it rent per week if the store increased its advertising budget to $700 and raised its rental price to $3.50?

46. **Stopping Distance of a Car** The stopping distance, $d$, of a car after the brakes are applied varies directly as the square of the speed, $s$, of the car. If a car traveling at a speed of 40 mph can stop in 80 ft, what is the stopping distance of a car traveling at 65 mph?

47. **Guitar Strings** The number of vibrations per second, $v$, of a guitar string varies directly as the square root of the tension, $t$, and inversely as the length of the string, $l$. If the number of vibrations per second is 5 when the tension is 225 kg and the length of the string is 0.60 m, determine the number of vibrations per second when the tension is 196 kg and the length of the string is 0.70 m.

48. **Electrical Resistance** The electrical resistance of a wire, $R$, varies directly as its length, $L$, and inversely as its cross-sectional area, $A$. If the resistance of a wire is 0.2 ohm when the length is 200 ft and its cross-sectional area is 0.05 in.$^2$, what is the resistance of a wire whose length is 5000 ft with a cross-sectional area of 0.01 in.$^2$?

49. **Phone Calls** The number of phone calls between two cities during a given time period, $N$, varies directly as the populations $p_1$ and $p_2$ of the two cities and inversely to the distance, $d$, between them. If 100,000 calls are made between two cities 300 mi apart and the populations of the cities are 60,000 and 200,000, how many calls are made between two cities with populations of 125,000 and 175,000 that are 450 mi apart?

50. **Direct and Inverse Variation**
   a) If $y$ varies directly as $x$ and the constant of proportionality is 2, does $x$ vary directly or inversely as $y$? Explain.
   b) Give the new constant of proportionality for $x$ as a variation of $y$.

51. **Direct and Inverse Variation**
   a) If $y$ varies inversely as $x$ and the constant of proportionality is 0.3, does $x$ vary directly or inversely as $y$? Explain.
   b) Give the new constant of proportionality for $x$ as a variation of $y$.

**Challenge Problems/Group Activities**

52. **Photography** An article in the magazine *Outdoor and Travel Photography* states, “If a surface is illuminated by a point-source of light, the intensity of illumination produced is inversely proportional to the square of the distance separating them. In practical terms, this means that foreground objects will be grossly overexposed if your background subject is properly exposed with a flash. Thus direct flash will not offer pleasing results if there are any intervening objects between the foreground and the subject.”

   If the subject you are photographing is 4 ft from the flash and the illumination on this subject is 1200 lux, what is the intensity of illumination on an intervening object that is 3 ft from the flash?

53. **Water Cost** In a specific region of the country, the amount of a customer’s water bill, $W$, is directly proportional to the average daily temperature for the month, $T$, the lawn area,
A, and the square root of $F$, where $F$ is the family size, and inversely proportional to the number of inches of rain, $R$.

In one month, the average daily temperature is 78°F and the number of inches of rain is 5.6. If the average family of four who has a thousand square feet of lawn pays $72.00 for water for that month, estimate the water bill in the same month for the average family of six who has 1500 ft$^2$ of lawn.

4.2 LINEAR INEQUALITIES

Suppose your campus bookstore needs to determine how many textbooks for a particular course must be sold so that the bookstore’s textbook revenue is greater than its cost. To determine this number, an inequality would be set up. In this section, we will discuss how to set up and how to solve an inequality.

The symbols of inequality are as follows.

**SYMBOLS OF INEQUALITY**

- $a < b$ means that $a$ is less than $b$.
- $a \leq b$ means that $a$ is less than or equal to $b$.
- $a > b$ means that $a$ is greater than $b$.
- $a \geq b$ means that $a$ is greater than or equal to $b$.

An *inequality* consists of two (or more) expressions joined by an inequality sign.

**Examples of inequalities**

- $3 < 5$
- $x < 2$
- $3x - 2 \geq 5$

A statement of inequality can be used to indicate a set of real numbers. For example, $x < 2$ represents the set of all real numbers less than 2. Listing all these numbers is impossible, but some are $-2$, $-1.234$, $-1$, $-\frac{1}{2}$, $0$, $\frac{9}{16}$, 1.

To indicate all real numbers less than 2, we can use the number line. The number line was discussed in Chapter 1.

**Solving Inequalities**

To indicate the solution set of $x < 2$ on the number line, we draw an open circle at 2 and a line to the left of 2 with an arrow at its end. This technique indicates that all points to the left of 2 are part of the solution set. The open circle indicates that the solution set does not include the number 2.

\[
\begin{array}{c}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\[x < 2\]

To indicate the solution set of $x \leq 2$ on the number line, we draw a closed (or darkened) circle at 2 and a line to the left of 2 with an arrow at its end. The closed circle indicates that the 2 is part of the solution.

\[
\begin{array}{c}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\[x \leq 2\]